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Correlation of the Solution of Two-Dimensional Steady State Heat Conduction by Analytical and Finite Element Approach

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ABSTRACT: *Heat transfer is an important problem in many disciplines including science and engineering. The present work focuses on analytical and numerical approaches to solve two-dimensional steady-state heat conduction problems. It can be seen from the literature, rigorous analytical solutions are available only for a very few simple boundary conditions and these are not amenable for complex boundaries. However, majority of engineering problems with simple as well as complex boundary conditions can be solved by using numerical approaches finite element method being the most effective one. In this work, an attempt is made to obtain solutions using finite element method for some typical two-dimensional steady state heat conduction problems. The results obtained by finite element approach which are in good agreement with the analytical values for some bench mark problems reveal that FEM can be effectively used solve more complex thermal problems also.*
Keywords *- Analytical Approach, Boundary conditions, Heat transfer, Numerical Approach, Steady State Heat Conduction*

I. INTRODUCTION

Heat transfer is an important problem in many disciplines including science and engineering. Heat transfer is defined as energy in transit. Analysis of a system using the laws of heat transfer is named as thermal analysis. Heat transfer is a branch of thermodynamics which deals with rate of heat transfer between two or more equilibrium states of a system in a medium or between media. In several situations one dimensional conduction approximation provides reasonably acceptable answers. However there are situations where the heat conduction in two dimensions has to be considered. One example is corners in a rectangular furnace. The solution required is the temperature at various locations and the heat flow. If the temperature field is established, heat flow can be determined by Fourier conduction equation.

Tian et al [1] has adopted the hybrid numerical method to investigate the heat conduction of functionally graded materials (FGM) plates with variable gradient parameters under the exponential heat source load. They reported that, the change in temperature at different position of the FGM plate is consistent with heat source. The distribution of the temperature is gradually reducing with the distance away from the heat source, and it tends to zero in infinite distance position. The change of the gradient parameter in a certain range has a great influence on the heat conduction. Jiang et al [2] discussed the analytical solutions for three-dimensional steady and transient heat conduction problems of a double-layer plate with a local heat source. Results of their investigation reveal that, the heat conduction is significantly influence by coating material, local heat source and structural parameters on temperature distribution of the double-layer plate. Liu et al [3] reveal that the analytical result has showed a good agreement with the results obtained by FEM for a two-dimensional heat conduction problem. Dobbertean et al [4] has adopted numerical approach for solving steady state heat transfer for jet impingement on patterned surfaces. They observed that the shape of the patterned surface (rectangular, triangular) does have a significant influence on the heat transfer coefficient.

Mainly four methods have been in use in solving two dimensional steady state heat conduction problems. These are;

A. Analytical approach
B. Graphical approach
C. Experimental approach
D. Numerical approach

While analytical solutions are available only for very simple boundary conditions, these are not applicable for complex boundaries. However, 2D thermal problems with simple as well as complex boundary conditions can be solved by using numerical approaches such as boundary element method (BEM), finite volume method (FVM), finite difference method (FDM) and finite element method (FEM). Finite element method is most popular numerical method to solve linear, non-linear, buckling, thermal, dynamic, fatigue and various problems. Hence, the present work explores the possibility of using finite element method to solve two-dimensional steady state heat conduction problems of rectangular plates with different boundary conditions.

II. ANALYTICAL APPROACH

To analyse the steady state two-dimensional heat transfer by conduction with no heat generation [5], the Laplace equation can be used which is given by (1),

$$\frac{d^2T}{dx^2} + \frac{d^2T}{dy^2} = 0 \qquad (1)$$

By assuming constant thermal conductivity, the solution to the equation (1) may be obtained by analytical, numerical, or graphical techniques. The objective of any heat-transfer analysis is usually to predict heat flow or the temperature that results from a certain heat flow. The solution to equation (1) will give the temperature in a two-dimensional body as a function of the two independent space coordinates x and y. then the heat flow in the x and y directions may be calculated from the Fourier equations (2) and (3). It is possible to establish the heat flow, if the temperature in the distribution in the material is known.

$$q_x = -k\,A_x \frac{\partial T}{\partial x} \qquad (2)$$

$$q_y = -k\,A_y \frac{\partial T}{\partial y} \qquad (3)$$

These heat-flow quantities are directed either in the x-direction or in the y-direction. The total heat flow at any point in the material is the resultant of the q_x and q_y at that point. Thus the total heat-flow vector is directed so that it is perpendicular to the lines of constant temperature in the material, as shown in Figure 1.

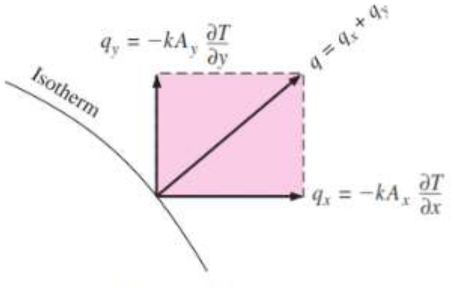

Figure 1. Heat flow in two-directions

To solve equation (1), the separation-of-variable method is used. The essential point of this method is that the solution to the differential equation is assumed to take the product form,

$$\left. \begin{aligned} T = XY \quad \text{where} \quad & X=X(x) \\ & Y=Y(y) \end{aligned} \right\} \qquad (4)$$

The boundary conditions are then applied to determine the form of the function X and Y.

$$\left. \begin{aligned} \frac{dT}{dx} &= X.\frac{dX}{dy} \;, \quad \frac{dT}{dx} = Y.\frac{dX}{dx} \\ \frac{d^2T}{dx^2} &= Y.\frac{d^2X}{dx^2} \;, \quad \frac{d^2T}{dy^2} = X.\frac{d^2Y}{dy^2} \end{aligned} \right\} (5)$$

Substituting equation (5) in (1) gives,

$$-\frac{1}{X}\frac{d^2X}{dx^2} = \frac{1}{Y}\frac{d^2Y}{dy^2} \qquad (6)$$

It can be seen that, each side of equation (6) is independent of each other because x and y are independent variables. This requires that each side be equal to some constant. Thus, two differential equations are expressed in terms of this constant

$$-\frac{1}{X}\frac{d^2X}{dx^2} = \frac{1}{Y}\frac{d^2Y}{dy^2} = \lambda^2 \qquad (7)$$

Further, this equation gives two differential equations,

$$\frac{d^2X}{dx^2} + \lambda^2 X = 0 \qquad (8)$$

$$\frac{d^2Y}{dy^2} - \lambda^2 Y = 0 \qquad (9)$$

Where λ^2 is called the separation constant. Its value must be determined from the boundary conditions. Consider the rectangular plate shown in Figure 2 with three sides of the plate is maintained at the constant temperature T_1 , and the upper side has some temperature distribution impressed upon it. This distribution could be simply a constant temperature or something more complex, such as a sine-wave distribution [5].

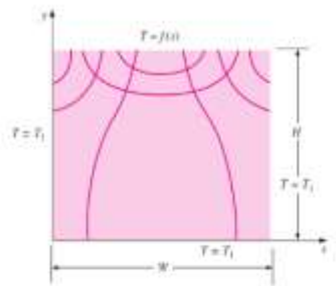

Figure 2. Isotherms and heat flow lines in a rectangular plate.

The boundary conditions are,

$$\left.\begin{array}{ll} T = T_1 & \text{at } y = 0 \\ T = T_1 & \text{at } x = 0 \\ T = T_1 & \text{at } x = W \\ T = T_m sin\left(\frac{\pi x}{w}\right) + T_1 & \text{at } y = H \end{array}\right\} \quad (10)$$

Where Tm is the amplitude of the sine function.

The first step in solving the problem is examine the value of (λ^2). There are three possibilities (λ^2=0, λ^2 <0, and λ^2 >0) one of this value is acceptable, and the other will be neglected.
For $\lambda^2 = 0$;

$$\frac{d^2X}{dx^2} = 0 \quad \text{and} \quad \frac{d^2Y}{dy^2} = 0$$

The solutions are,

$$\left.\begin{array}{l} X = C_1 + C_2 x \\ Y = C_3 + C_4 y \\ T = (C_1 + C_2 x)(C_3 + C_4 y) \end{array}\right\} (11)$$

This function cannot fit the sin-function boundary condition, so that the $\lambda^2 = 0$ solution may be excluded.

For $\lambda^2 < 0$;

The solutions are,

$$\left.\begin{array}{l} X = C_5 e^{-\lambda x} + C_6 e^{\lambda x} \\ Y = C_7 \cos\lambda y + C_8 \sin\lambda y \\ T = (C_5 e^{-\lambda x} + C_6 e^{\lambda x})(C_7 \cos\lambda y + C_8 \sin\lambda y) \end{array}\right\} (12)$$

Again, the sine-function boundary condition cannot be satisfied, so this solution is excluded.

For $\lambda^2 > 0$;

The solutions are,

$$\left.\begin{array}{l} X = C_9 \cos\lambda x + C_{10} \sin\lambda x \\ Y = C_{11} e^{-\lambda y} + C_{12} e^{\lambda y} \\ T = (C_9 \cos\lambda x + C_{10} \sin\lambda x)(C_{11} e^{-\lambda y} + C_{12} e^{\lambda y}) \end{array}\right\} (13)$$

Now, it is possible to satisfy the sin-function boundary condition; hence, it is essential to satisfy the other conditions. The new variable is θ,

$$\theta = T - T_1$$

It is required to transform the boundary conditions to the corresponding value of ,

$$\left.\begin{array}{ll} \theta = 0 & \text{at } y = 0 \\ \theta = 0 & \text{at } x = 0 \\ \theta = 0 & \text{at } x = W \\ \theta = T_m sin\left(\frac{\pi x}{w}\right) & \text{at } y = H \end{array}\right\} (14)$$

Applying these conditions, we get;

$$0 = (C_9 \cos\lambda x + C_{10} \sin\lambda x)(C_{11} + C_{12}) \quad [a]$$

$$0 = C_9(C_{11} e^{-\lambda y} + C_{12} e^{\lambda y}) \quad [b]$$

$$0 = (C_9 \cos\lambda W + C_{10} \sin\lambda W)(C_{11} e^{-\lambda y} + C_{12} e^{\lambda y}) \quad [c]$$

$$T_m \sin\left(\frac{\pi x}{W}\right) = (C_9 \cos\lambda x + C_{10} \sin\lambda x)(C_{11} e^{-\lambda H} + C_{12} e^{\lambda H}) \quad [d]$$

Accordingly,

$C_{11} = - C_{12}$

$C_9 = 0$

and from [c],

$$0 = C_{10} C_{12} \sin\lambda W \left(e^{\lambda y} - e^{-\lambda y}\right)$$

This requires that, sin W = 0 (15)

 is an undetermined separation constant.

$$\lambda = \frac{n\pi}{W} \quad (16)$$

Where n is an integer. The solution to the differential equation may thus be written as a sum of the solutions for each value of n. This is an infinite sum, so that the final solution is the infinite series.

$$\theta = T - T_1 = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{W}\right) \sinh\left(\frac{n\pi y}{W}\right) \quad (17)$$

Where the constant have combined and the exponential terms converted to the hyperbolic function. The final boundary condition may now be applied;

$$T_m \sin\left(\frac{\pi x}{W}\right) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{W}\right) \sinh\left(\frac{n\pi H}{W}\right)$$

Which requires that Cn = 0 for n >1. Hence, the final solution for this 2D thermal condition is:

$$T_{(x,y)} = T_1 + T_m \frac{\sinh\left(\frac{\pi y}{W}\right)}{\sinh\left(\frac{\pi H}{W}\right)} \sin\left(\frac{\pi x}{W}\right) \quad (18)$$

Now, consider the second set of boundary conditions with simply a constant temperature on the edge of the plate. Thus

$T = T_1$	at	$y = 0$
$T = T_1$	at	$x = 0$
$T = T_1$	at	$x = W$
$T = T_2$	at	$y = H$

Using the first three boundary conditions, we obtain the solution in the form of equation (15);

$$T - T_1 = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{W}\right) \sinh\left(\frac{n\pi y}{W}\right) \quad (19)$$

Applying the fourth boundary condition gives,

$$T_2 - T_1 = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{W}\right) \sinh\left(\frac{n\pi H}{W}\right) \quad (20)$$

This is a Fourier sine series, and the value of the Cn can be determined by expanding the constant temperature difference T2 – T1 in a Fourier series over the interval 0<x <W. This series is,

$$T_2 - T_1 = (T_2 - T_1)\, \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{(n+1)} + 1}{n} \sin\left(\frac{n\pi x}{W}\right) \quad (21)$$

Upon comparison of equation (20) with equation (21), we find that,

$$C_n = \frac{2}{\pi}\,(T_2 - T_1)\, \frac{1}{\sinh\left(\frac{n\pi H}{W}\right)}\, \frac{(-1)^{(n+1)} + 1}{n}$$

and the final solution for this 2D thermal condition is:

$$\frac{T - T_1}{T_2 - T_1} = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{(n+1)} + 1}{n} \sin\left(\frac{n\pi x}{W}\right) \frac{\sinh\left(\frac{n\pi y}{W}\right)}{\sinh\left(\frac{n\pi H}{W}\right)} \quad (22)$$

where n = 1, 3, 5…..

An extensive study of theoretical (analytical) technique used in conduction heat transfer requires a background in the theory of orthogonal functions. Fourier series are one example of orthogonal functions, as are Bessel function and other special functions applicable to different geometries and boundary conditions.

III. FINITE ELEMENT APPROACH

Finite element analysis (FEA) or finite element method (FEM) is a numerical analysis technique for obtaining approximate solutions to a wide variety of engineering problems. It is probably the best approach to solve 2D and / or 3D thermal problems. Finite element formulation for heat transfer problems can be derived either by using variational methods or by using weighted-residual methods.

The element conduction matrix [K_C] can be derived from the equation (23);

$$[K_c] = \int_0^v [B]^T\,[D]\,[B]\,dv \quad (23)$$

Where, D is a thermal conductivity matrix, and can be expressed for 2D thermal problem as,

$$D = \begin{bmatrix} K_{xx} & 0 \\ 0 & K_{yy} \end{bmatrix} \quad (24)$$

Kxx , Kyy is the thermal conductivity of the material along x-axis, y-axis respectively.

[B] is derivative of shape functions [N]. Now to determine the matrix [B], it is essential to consider 2D element. In the present work, a triangular element (Figure 3) having 3 nodes have been considered for FEA. For this element, the temperature [T] function could be expressed as,

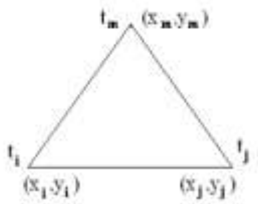

Figure 3. Triangular Element

$$T = N_i t_i + N_j t_j + N_m t_m \quad (25)$$

In the matrix form, equation (25) could be written as,

$$[T]=\begin{bmatrix} N_i & N_j & N_m \end{bmatrix}\begin{Bmatrix} t_i \\ t_j \\ t_m \end{Bmatrix} \qquad (26)$$

Where,

ti, tj, tm are the nodal temperature at the nodes i, j, k respectively.

Ni, Nj, Nm are the shape functions at the nodes i, j, k respectively.

Now, the temperature gradient relationship could be expressed as,

$$\{g\}=\begin{Bmatrix} \dfrac{\partial T}{\partial x} \\ \dfrac{\partial T}{\partial y} \end{Bmatrix}=\begin{bmatrix} \dfrac{\partial N_i}{\partial x} & \dfrac{\partial N_j}{\partial x} & \dfrac{\partial N_m}{\partial x} \\ \dfrac{\partial N_i}{\partial y} & \dfrac{\partial N_j}{\partial y} & \dfrac{\partial N_m}{\partial y} \end{bmatrix}\begin{Bmatrix} t_i \\ t_j \\ t_m \end{Bmatrix}$$

Analogous to strain matrix: {g} = [B]{t}

$$[B]=\frac{\partial}{\partial x}[N]=\begin{bmatrix} \dfrac{\partial N_i}{\partial x} & \dfrac{\partial N_j}{\partial x} & \dfrac{\partial N_m}{\partial x} \\ \dfrac{\partial N_i}{\partial y} & \dfrac{\partial N_j}{\partial y} & \dfrac{\partial N_m}{\partial y} \end{bmatrix}=\frac{1}{2A}\begin{bmatrix} \beta_i & \beta_j & \beta_m \\ \gamma_i & \gamma_j & \gamma_m \end{bmatrix} \qquad (27)$$

$$\text{Where, } N_i=\frac{1}{2A}\left[\alpha_i+\beta_i x+\gamma_i y\right]$$

$$N_j=\frac{1}{2A}\left[\alpha_j+\beta_j x+\gamma_j y\right]$$

$$N_m=\frac{1}{2A}\left[\alpha_m+\beta_m x+\gamma_m y\right]$$

$$\text{and } \alpha_i=x_j y_m-y_j x_m,\ \beta_i=y_j-y_m,\ \gamma_i=x_m-x_j$$

$$\alpha_j=x_m y_i-y_m x_i,\ \beta_j=y_m-y_i,\ \gamma_j=x_i-x_m$$

$$\alpha_m=x_i y_j-y_i x_j,\ \beta_m=y_i-y_j,\ \gamma_m=x_j-x_i$$

Substitute [B] matrix and [D] matrix in equation (23),

$$[K_c]=\int_V \frac{1}{2A^2}\begin{bmatrix} \beta_i & \gamma_i \\ \beta_j & \gamma_j \\ \beta_m & \gamma_m \end{bmatrix}\begin{bmatrix} K_{xx} & 0 \\ 0 & K_{yy} \end{bmatrix}\begin{bmatrix} \beta_i & \beta_j & \beta_m \\ \gamma_i & \gamma_j & \gamma_m \end{bmatrix} dV$$

If thickness is assumed constant and all terms of integrand constant, then the element conduction matrix is,

$$[K_c] = [B]^T\,[D]\,[B]\;t\,A \qquad (28)$$

Where, t is the thickness of the plate
A is the area

If it is required to consider convection contribution, then the convection matrix [Kh] because of the side exposed to free stream temperature can be derived from the equation (29),

$$[K_h] = \int_0^s h\,[N]^T\,[N]\,ds \qquad (29)$$

$$[K_h]=h\int_S \begin{bmatrix} N_iN_i & N_iN_j & N_iN_m \\ N_jN_i & N_jN_j & N_jN_m \\ N_mN_i & N_mN_j & N_mN_m \end{bmatrix} dS$$

Hence, the element convection matrix is,

$$[K_h] = \frac{h\,L_{(i-j)}\,t}{6}\begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad (30)$$

If heat source Q is acting uniformly over each element, then the element heat generator vector for an element is given by the equation (31),

$$\{f^e\} = \frac{Q\,v}{3}\begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix} \qquad (31)$$

Where v is the volume of 2D object.

If no heat generation Q = 0 and q* = 0, then the convective vector for an element is given by,

$$\{f^e\} = \begin{Bmatrix} f_2 \\ f_3 \\ f_5 \end{Bmatrix} = \frac{hT_\infty\,L_{(i-j)}\,t}{2}\begin{Bmatrix} 1 \\ 1 \\ 0 \end{Bmatrix} \qquad (32)$$

IV. STEADY STATE HEAT CONDUCTION

For the given boundary conditions as shown in Figure 4, determine the temperature at the midpoint P of the plate under steady two dimensional conduction.

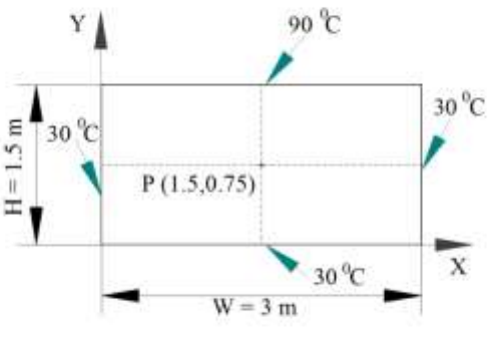

Figure 4. 2D steady state thermal problem with boundary condition

A) Analytical Solution

The series solution for this 2D thermal condition is:

$$\frac{T_{(x,y)}-T_1}{T_2-T_1} = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{(n+1)}+1}{n}\sin\left(\frac{n\,\pi\,x}{W}\right)\frac{\sinh\left(\frac{n\,\pi\,y}{W}\right)}{\sinh\left(\frac{n\,\pi\,H}{W}\right)}$$

where n = 1, 3, 5…..

Given:

$T_1 = 30\ ^0C$

$T_2 = 90\ ^0C$

W = 3 m, H = 1.5 m

P(x,y) = (1.5, 0.75)

Hence,

$$\frac{x}{W} = \frac{1.5}{3} = 0.5$$

$$\frac{y}{W} = \frac{0.75}{3} = 0.25$$

$$\frac{H}{W} = \frac{1.5}{3} = 0.5$$

In this problem, we have used up to five terms in the series summation.

$$\frac{T_{(1.5,0.75)}-30}{90-30} = \frac{2}{\pi}\left[2\sin\left(\frac{\pi}{2}\right)\frac{\sinh\left(\frac{\pi}{4}\right)}{\sinh\left(\frac{\pi}{2}\right)} + \frac{2}{3}\sin(1.5\,\pi)\frac{\sinh(0.75\,\pi)}{\sinh(1.5\,\pi)} + \frac{2}{5}\sin(2.5\,\pi)\frac{\sinh(1.25\,\pi)}{\sinh(2.5\,\pi)} + \frac{2}{7}\sin(3.5\,\pi)\frac{\sinh(1.75\,\pi)}{\sinh(3.5\,\pi)} + \frac{2}{9}\sin(4.5\,\pi)\frac{\sinh(2.25\,\pi)}{\sinh(4.5\,\pi)}\right]$$

$$\frac{T_{(1.5,0.75)}-30}{90-30} = \frac{2}{\pi}[0.7549 - 0.0626 + 0.0078 - 0.0012 + 0.0002]$$

$$\frac{T_{(1.5,0.75)}-30}{90-30} = 0.4451$$

Therefore, the temperature at the point P (1.5, 0.75) is found to be 56.71 ^0C.

B) Finite Element Approach

The analysis by finite element method (FEM) is accomplished by using ANSYS software. In the present work, for steady state heat conduction, PLANE55 element (2-D thermal solid element) shown in Figure 5 was used to develop finite element model for thermal analysis. PLANE55 can be used as a plane element or as an axi-symmetric ring element with a 2-D thermal conduction capability. The quadrilateral element has four nodes with a single degree of freedom, temperature, at each node. Plane 55 also have a triangular option for the shape of an element (Figure 5). This element is applicable to a 2-D, steady-state or transient thermal analysis. The element can also compensate for mass transport heat flow from a constant velocity field.

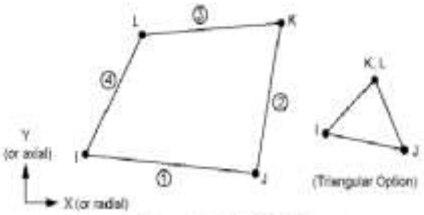

Figure 5. PLANE55 Element

By using finite element approach, the continuum (rectangular plate) is discritized in to number of PLANE55 triangular elements as shown in Figure 6. This is followed by incorporating the material properties like thermal conductivity and then applying the boundary conditions as detailed in Figure 4.

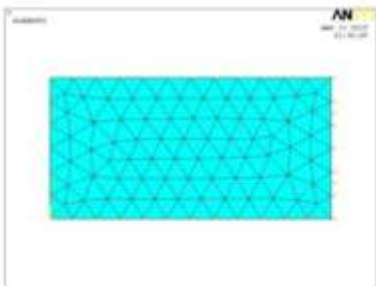

Figure 6. Descritized rectangular plate

After solving, it was observed that the temperature at the location of interest P (1.5, 0.75) was found to be 56.66 ^0C. The results obtained from finite element approach using Ansys software shown in Figure 7 is in good agreement with the analytical solutions.

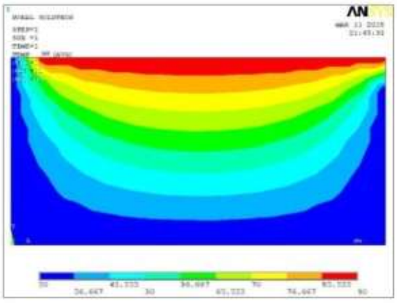

Figure 7. Solution obtained from finite element approach using Ansys software

Further, the feasibility of finite element approach is explored to solve another 2D heat transfer problem with conduction, convection and insulation boundary condition for a two-dimensional body as shown in Figure 8.

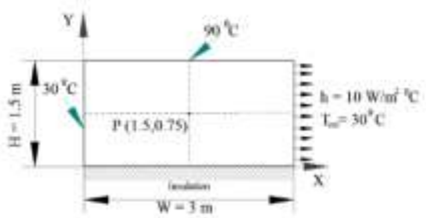

Figure 8. 2D heat transfer problem with conduction, convection and insulation boundary conditions

The rectangular plate with 3m width and 1.5m height is discritized in to number of PLANE55 triangular elements similar to the previous problem shown in Figure 6. However, this 2D heat transfer problem with conduction, convection and insulation boundary conditions are incorporated as detailed in Figure 8. After solving the 2D heat transfer problem with complex boundary conditions, it was observed that the temperature at the location of interest P (1.5, 0.75) was found to be 74.702 ^{0}C. The results obtained from finite element approach using Ansys software is shown in Figure 9.

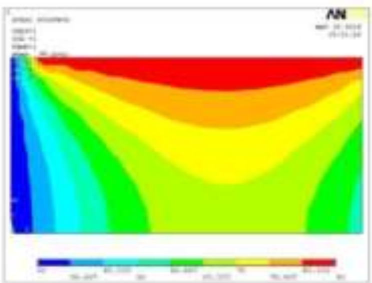

Figure 9. Solution obtained from finite element approach using Ansys software

V. CONCLUSION

Rigorous analytical solutions are applicable only for very simple boundary conditions and these are not amenable for complex boundaries. However, 2D thermal problems with simple as well as complex boundary conditions can be solved by using numerical approach. In the present work, finite element method (numerical approach) was used to solve 2D steady state heat transfer problems. Based on the results obtained from the present study, the following important conclusion are drawn,

1. For a steady state heat conduction problem, it was observed that the temperature at the location of interest P (1.5, 0.75) was found to be 56.71 ^{0}C by analytical approach and 56.66 ^{0}C by finite element approach (numerical approach). Results reveal that, solutions from finite element approach are in good agreement with the solutions obtained by analytical approach.
2. With the same finite element approach, solutions for a steady state heat conduction problem with complex boundary conditions such as insulation and convection, the temperature at the point P (1.5, 0.75) was found to be 74.702 ^{0}C in a rectangular plate.
3. Finite element approach is a very simple technique and is used to solve 2D steady state heat transfer problems with simple and complex boundary conditions.

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